Preparation Materials
FOR
IPhO

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NePhO

Nepal Physical Society

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[2]Source: Problems in General Physics, I.E. Irodov

[3]Source: Griffiths, David Jeffrey, and Reed College. Introduction to electrodynamics. Vol. 3. Upper Saddle River, NJ: prentice Hall, 1999.

[4]This section is taken from Schutz, Bernard. A first course in general relativity. Cambridge university press, 2009.

Chapter 1

Preface

I started this note when I was involved in training the 20 selected students from the first round of NePhO, 2015. Being a past IPhO participant, I am highly interested in thinking and designing of interesting problems of the IPhO level. Problem solving requires a good deal of background knowledge, and more importantly the ability to visualise the problem situation properly and think critically. I have tried to present some key knowledge and skills in this short note. This note is by no means comprehensive and complete, and I will try to add on more to this note in the future. If you have any comments, questions and suggestions, you could always feel free to reach me though email: dnshkandel@gmail.com.

Dinesh Kandel
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Chapter 2

Problem Solving Techniques

2.1 Hints on Solving Physics Problems[1]

1. First of all, look through the reference datas given to you, for many problems cannot be solved without them. Besides, the reference data will make your work easier and save your time.

2. Begin the problem by recognizing its meaning and its formulation. Make sure that the data given are sufficient for solving the problem. Wherever possible, draw a diagram elucidating the essence of the problem; in many cases this simplifies both the search for a solution and the solution itself.

3. Solve each problem, as a rule, in the general form, that is in a letter notation, so that the quantity sought will be expressed in the same terms as the given data. A solution in the general form is particularly valuable since it makes clear the relationship between the sought quantity and the given data. What is more, an answer obtained in the general form allows one to make a fairly accurate judgement on the correctness of the solution itself (see the next item).

4. Having obtained the solution in the general form, check to see if it has the right dimensions. The wrong dimensions are an obvious indication of a wrong solution. If possible, investigate the behaviour of the solution in some extreme special cases. For example, whatever the form of the expression for the gravitational force between two extended bodies, it must turn into the well-known law of gravitational interaction of mass points as the distance between the bodies increases. Otherwise, it can be immediately inferred that the solution is wrong.

5. When starting calculations, remember that the numerical values of physical quantities are always known only approximately. Therefore, in calculations you should employ the rules for operating with approximate

[1]Source: Problems in General Physics, I.E. Irodov

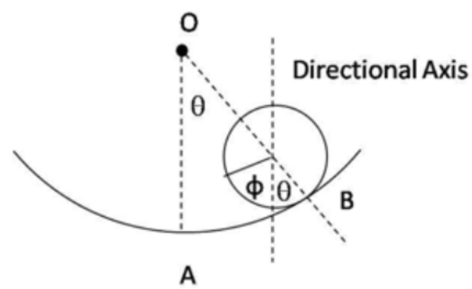

Figure 2.1: Displaced position of the rolling cylinder.

numbers. In particular, in presenting the quantitative data and answers strict attention should be paid to the rules of approximation and numerical accuracy.

6. Having obtained the numerical answer, evaluate its plausibility. In some cases such an evaluation may disclose an error in the result obtained. For example, a stone cannot be thrown by a man over the distance of the order of 1 km, the velocity of a body cannot surpass that of light in a vacuum, etc.

2.2 Energy Conservation: A powerful Problem Solving Technique

2.2.1 Problem: Rolling Cylinder Inside A Fixed Tube

A cylinder of radius r and mass M is rolling perfectly inside a tube of radius R. Find the frequency of small oscillation inside the tube.

Approach: There are two ways of attacking this problem. First is taking into consideration forces and torques, while the second is applying energy conservation. We will see the that the second method is much easier to implement.

There are two constraints on the cylinder. (1) It should roll inside the tube. (2) It should roll. Note that the angluar velocity of the cylinder is $\omega = -\dot{\theta}\hat{z}$. There exits relationship between $\dot{\theta}$ and $\dot{\phi}$ because of the rolling condition:

$$R\theta = r(\theta + \phi), \tag{2.1}$$

which upon differentiation gives

$$\dot{\phi} = \frac{(R-r)}{r}\dot{\theta}. \tag{2.2}$$

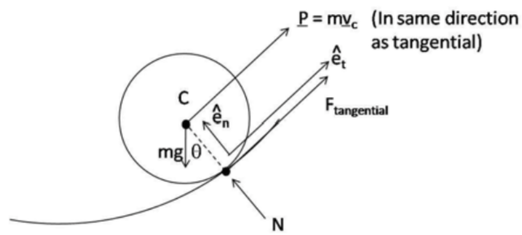

Figure 2.2: Forces involved in the problem.

2.2.1.1 Force Method

Consider the angular momentum of rolling cylinder about its center of mass. As the center of mass velocity $\mathbf{v_c}$ is parallel to the center of mass momentum $\mathbf{P}$,

$$\mathbf{v_c} \times \mathbf{P} = 0, \tag{2.3}$$

where bold face implies that the quantity is vector. As the torque through center mass τ_c is

$$\tau_c = \frac{d\mathbf{H_c}}{dt} = -rF\hat{z}, \tag{2.4}$$

where F is due to friction and H_c is the angular momentum of the cylinder about its center of mass, given by

$$\mathbf{H_c} = I_c\omega = -\frac{1}{2}mr^2\frac{R-r}{r}\dot{\theta}\hat{z}. \tag{2.5}$$

Using this equation in equation (13) gives

$$rF = \frac{1}{2}mr^2\frac{R-r}{r}\ddot{\theta} \tag{2.6}$$

To find F, we use conservation of linear momentum (ref 2.2).

$$\begin{aligned}(F - mg\sin\theta)\hat{e}_t &= \frac{d}{dt}(mv_c\hat{e}_t) \\ &= \frac{d}{dt}(m\dot{\theta}(R-r)\hat{e}_t) \\ &= m\ddot{\theta}(R-r)\hat{e}_t + m\dot{\theta}(R-r)\dot{\hat{e}}_t \\ &= m\ddot{\theta}(R-r)\hat{e}_t + m\dot{\theta}^2(R-r)\hat{e}_n,\end{aligned} \tag{2.7}$$

where in the last step we have used the fact that $\dot{\hat{e}}_t = \dot{\theta}\hat{e}_n$. It can also be seen from (16) that $m\dot{\theta}^2(R-r) = 0$, as there is no normal component vector on the left side of that equation. Finally, comparing (15) and (16) gives

$$\boxed{(R-r)\ddot{\theta} + \frac{2}{3}g\sin\theta = 0.} \tag{2.8}$$

The frequency of oscillation will be found later in next sub-section by using small angle approximation.

2.2.1.2 Energy Consideration

This is the main highlight of the whole problem. The total energy of a system is $T+V$, where T is kinetic and V is potential energy. Note that although friction is present, as there is perfect rolling, the energy loss due to friction is essentially negligible for the situation we are interested in. The perfect rolling condition implies

$$v_c = (R-r)\dot{\theta}, \tag{2.9}$$

where v_c is the velocity of the center of mass. Kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{R-r}{r}\dot{\theta}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot{\theta}^2. \end{aligned} \tag{2.10}$$

The potential energy of the system is

$$V = -mg(R-r)\cos\theta. \tag{2.11}$$

Using the fact that $\dot{E} = 0$, gives

$$\boxed{(R-r)\ddot{\theta} + \frac{2}{3}g\sin\theta = 0,} \tag{2.12}$$

which is same as the result of previous sub-section, but much more clear to compute. Finally, using small angle approximation $\sin\theta \approx \theta$ gives us

$$\omega = \sqrt{\frac{2}{3}\frac{g}{R-r}}. \tag{2.13}$$

2.3 Motions with non-constant mass

Consider the following problem (which appeared in the Selection test of NePhO, 2011): Let us model the physics of falling raindrop as the following: as the rain drop falls, its mass changes at a rate proportional to its surface area. Calculate the acceleration of the raindrop. Clearly state the assumptions you make, and write down how valid they are in a general atmospheric setting.

Let us think again about what information we are given in the problem. First, the raindrop is falling under the action of gravity. Since the problem says nothing about air-resistance, it is definitely legitimate to say that we ignore air-resistance. One can also solve the problem by assuming air drag is proportional to the speed of the raindrop, i.e. $\vec{F}_{\text{drag}} = -b\vec{v}$, where negative sign implies that the force is resistive. But this will yield a differential equation which might not be easily solvable for high school students. Therefore, we want to keep it simple: we ignore that the raindrop falls freely under the action of gravity. At this point, students should ask themselves: since the raindrop is falling freely, shouldn't the acceleration of the raindrop just be g? The answer is (as you might have already guessed): NO. This is because mass is changing, and therefore part of the gravitational force is compensated by the changing mass.

Let us now do some serious calculations. We start with the assumption that the raindrop is spherical, and has a constant density. Both the assumptions are good as sphere minimizes surface area, and the drop is made up of homogeneous water. We first write the force equation. Let $m(t)$ be the mass of the drop at any time t, and let the initial mass of the drop at $t = 0$ be $m(0) = 0$. Let $v(t)$ be the velocity of the drop at time t. Note that both $m(t)$ and $v(t)$ are changing over time. The net force acting on the spherical raindrop at any time t is $F = mg$ (I will ignore the directions, as it is pretty clear that the only relevant direction is vertically downwards). Using Newton's second law I can write:

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg \ , \tag{2.14}$$

which upon using product rule gives

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = mg \ , \tag{2.15}$$

or equivalently,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = g \ . \tag{2.16}$$

At this point, we use the fact that rate of change of mass is proportional to the surface area:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = c \times 4\pi r^2 \ , \tag{2.17}$$

where c is a proportionality constant. Using $m = \rho \times 4\pi r^3/3$, where ρ density of the drop and is constant, we get

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho \times 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}. \tag{2.18}$$

Using Eqs. 2.17 and 2.18, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{\rho} = \text{constant} \ , \tag{2.19}$$

which gives

$$r = \frac{c}{\rho}t \tag{2.20}$$

and therefore,

$$\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{1}{\rho \times 4\pi r^3/3}\rho \times 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{r}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{t} \ . \tag{2.21}$$

Using Eq.(2.16), we finally have

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3v}{t} = g \; . \tag{2.22}$$

First, it is always useful to check if the above equation is dimensionally correct (and it actually is). The above differential equation is not very easy to solve, therefore, you should already get most of the points just for deriving it. But, in order to fully solve it, you need the concept of integration factor. In this case the integration factor is $e^{\int \frac{3}{t}\mathrm{d}t} = t^3$. Multiplying both sides of Eq. (2.22) by this integration factor yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}(vt^3) = gt^3 \; , \tag{2.23}$$

which has the solution:

$$v(t) = \frac{gt}{4} \; . \tag{2.24}$$

Finally, the acceleration is just the first derivative of the velocity, and is therefore

$$\boxed{a(t) = \frac{g}{4} \; .} \tag{2.25}$$

As anticipated, the acceleration of the drop is less than g, as some part of the gravitational force is taken in changing mass. Makes sense, right? An important point to note is that the velocity changing linearly with time is not very sensible in the real life, otherwise raindrops falling from high up in the sky could potentially kill us. Usually, the gravitational force and drag force balance each other once the rain drop reaches some speed called *terminal speed*.

2.4 Gravity

Some of the most difficult problems in Physics Olympiad involves problems about gravity. Generally speaking, as a high school students you have already encountered important pieces: the concept of gravitational potential, gravitational energy, etc.

In this section, you will reinforce those concepts through a bit more challenging problems. Before that, you should keep in mind: **Gravitational force is central force, i.e., it acts only radially. Since torque is $\vec{\tau} = \vec{r} \times \vec{F}$, gravitational force does not produce any torque. Therefore, angular momentum of a gravitating body is *conserved*.** This statement is very powerful, and can help you solve a lot of problems involving gravity. Therefore, always keep in mind: *Angular momentum is conserved for an object which is only influenced by gravity.*

We start with a problem.

Problem

A satellite of mass m is designed to travel from earth to Planet X in an elliptical orbit. The perihelion of the satellite is at earth and aphelion at X. Assume that the earth and X have circular orbits around the *Sun* with radii R_E and R_X, respectively, and neglect the gravitational effects of the planets on the satellite, and take that the only gravitational force acting on the satellite is due to the Sun of mass M. (a) Find the velocity relative to earth, and the direction with which the satellite have to leave the earth in order to reach X. (b) How long will it take to reach X? (c) With what velocity relative to X, will it reach the orbit of X?

Solution: It is important to note that the orbit of satellite is elliptical, not circular. Therefore, you have to be careful with what radius r means each time. Firstly, the energy of the satellite should be conserved, and so should the angular momentum be. The kinetic energy of satellite is sum of circular kinetic energy and radial kinetic energy, as both radius and angle changes during its elliptical motion. Therefore,

$$K.E = \frac{1}{2}mv_r^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv_r^2 + \frac{1}{2}mr^2\omega^2 = \frac{1}{2}mv_r^2 + \frac{1}{2}\frac{L}{mr^2} \ ,$$

where v_r is the radial velocity, $L = I\omega = mr^2\omega$ =constant is the angular momentum of the satellite. The potential energy of the satellite is purely due to the gravitational attraction of the Sun. Therefore, it is $-GMm/r$. Thus, the total energy is

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}\frac{L}{r^2} - \frac{GMm}{r} \ . \tag{2.26}$$

At the perihelion and aphelion, $v_r = 0$ and $r = R_E$ and $r = R_X$ respectively. Thus, energy conservation gives

$$E = \frac{1}{2}\frac{L^2}{mR_E^2} - \frac{GMm}{R_E} = \frac{1}{2}\frac{L^2}{mR_X^2} - \frac{GMm}{R_X} \ , \tag{2.27}$$

which gives

$$L = \sqrt{\frac{2GMm^2R_ER_X}{R_E + R_X}} \ . \tag{2.28}$$

At perihelion, the velocity (purely due to rotation motion) is

$$v = \omega r = \frac{L}{mR_E} = \sqrt{\frac{2GMR_X}{R_E(R_E + R_X)}} \ . \tag{2.29}$$

We would like to minimise the speed with which the satellite has to leave the earth, and that would occur if the satellite is launched in a direction parallel to the earth's revolution around the sun, as the orbital motion it already has would act as an additional kick. Therefore, the velocity with which the satellite should leave the earth is

$$v_r = v - v_E = \sqrt{\frac{2GMR_X}{R_E(R_E + R_X)}} - \sqrt{\frac{GM}{R_E}} \ , \tag{2.30}$$

and in the same way, the velocity it should have at the aphelion is

$$v_r' = v' - v_M = \sqrt{\frac{2GMR_E}{R_M(R_E + R_X)}} - \sqrt{\frac{GM}{R_M}} \ . \tag{2.31}$$

From Kepler's third law, the period T of the revolution of the Satellite around the sun is

$$T^2 = T_E^2 R_E^{-3} \left(\frac{R_E + R_M}{2} \right)^3 \ , \tag{2.32}$$

where T_E is the period of revolution of earth which is equal to 1 year. Therefore, the period for the Satellite to reach X is

$$t = \frac{T}{2} = \frac{1}{2} \left(\frac{R_E + R_M}{2} \right)^{3/2} \ \text{years} \ . \tag{2.33}$$

Chapter 3

Miscelleneous Topics

3.1 Non-Intertial Frame of Reference

Newton's laws can only be applied to an inertial frame of reference. However, it can also be used in an inertial frame of reference if we include pseudo forces. This concept is illustrated in the following problem.

3.1.1 Elevating Pulley

A pulley of mass m_p has two masses m_1 and m_2 suspended on it with an elastic and massless rope. The pulley is itself suspended to the bottom of an elevator of mass m_e that is accelerating with acceleration a. Ignoring the diameter of the pulley, find out the accelerations of each block and tensions in each of the strings.
Solution:

Obviously, we have the gravitational forces on each object. The pulley also has $2T$ acting downward on it (due to the force exerted by the rope on the pulley) and R acting upward (the force pulling upward on the pulley through the rope connected to the elevator). Similarly, the elevator has tension forces R acting downward and E upward. We include the forces on the pulley and elevator since, a priori, its not obvious that they should be ignored. We will see that it is not necessary to solve for the forces on the pulley and elevator to find the accelerations of the masses, but we will be able to find these forces.

Coordinate system: Remember that Newtons second law only holds in inertial reference frames. Therefore, we should reference the positions of the masses to the fixed frame rather than to the elevator. Again, denote the z coordinates of the two masses by z_1 and z_2. Let the z coordinates of the pulley and elevator be z_p and z_e.

Equations of Motion:

$$\begin{aligned} m_1\ddot{z}_1 &= m_1 g + T \\ m_2\ddot{z}_2 &= -m_2 g + T \\ m_p\ddot{z}_p &= R - 2T - m_p g \\ m_e\ddot{z}_e &= E - R - m_e g. \end{aligned} \tag{3.1}$$

where T is the tension in the rope holding the two masses, R is the tension in the rope holding the pulley, and E is the force being exerted on the elevator to make it ascend or descend. Note especially the way we only consider the forces acting directly on an object; trying to unnecessarily account for forces is a common error. For example, even though gravity acts on m_1 and m_2 and some of that force is transmitted to and acts on the pulley, we do not directly include such forces; they are implicitly included by their effect on T. Similarly for the forces on the elevator.

Comstraints:the rope length cannot change, but the constraint is more complicated because the pulley can move:

$$z_1 + z_2 = 2z_p - l. \tag{3.2}$$

As the length of rope between the pulley and the elevator is fixed,

$$\ddot{z}_p = \ddot{z}_e = a, \tag{3.3}$$

so

$$\ddot{z}_1 + \ddot{z}_2 = 2a. \tag{3.4}$$

Using (3) and (4) in (1) will yield us the solution:

$$\begin{aligned} \ddot{z}_1 &= -\frac{m_1 - m_2}{m_1 + m_2} g + \frac{2m_2}{m_1 + m_2} a \\ \ddot{z}_2 &= \frac{m_1 - m_2}{m_1 + m_2} g + \frac{2m_1}{m_1 + m_2} a \\ T &= \frac{2m_1 m_2}{m_1 + m_2}. \end{aligned} \tag{3.5}$$

Note that $\ddot{z}_1 \neq -\ddot{z}_2$! But wait! Solving 6 simultaneous equation seems cumberson. Is there any clever and short method? Yes there is. Let us first calculate the accelerations of blocks in the reference frame of elevator. In this reference frame, the effective gravitational acceleration will be $g + a$, because this frame is not an intertial frame and to get correct results we need to include pseudo force ma. Let the accelerations in the frame of elevator of blocks be

$$\ddot{\tilde{z}}_1 = \ddot{z}_1 - a, \tag{3.6}$$

and

$$\ddot{\tilde{z}}_2 = \ddot{z}_2 - a \tag{3.7}$$

Now remember simple pulley problem's results. The accelerations are given by:

$$\begin{aligned}\ddot{z}_1 &= -\frac{m_1 - m_2}{m_1 + m_2}(g + a) \\ \ddot{z}_2 &= \frac{m_1 - m_2}{m_1 + m_2}(g + a).\end{aligned} \tag{3.8}$$

Using (8) in (6) and (7) will give us desired result (5). Solving for R and E using (1) gives

$$\begin{aligned}R &= \left(m_p + \frac{4m_1 m_2}{m_1 + m_2}\right)(g + a) \\ E &= \left(m_e + m_p + \frac{4m_1 m_2}{m_1 + m_2}\right)(g + a).\end{aligned} \tag{3.9}$$

3.2 Fermat's Principle

Fermats principle states that when a light ray travels between any two points, it takes the path with shortest time interval. Since the shortest distance between two points is a straight line, light rays travel in a straight line homogeneous medium.

In the following few steps, we will show how Fermats principle can be used to derive Snells law of refraction. Suppose that a light ray is to travel from point P in medium 1 to point Q in medium 2 as shown in Fig. (3.1) , where P and Q are at perpendicular distances a and b, respectively,from the interface. The speed of light is v_1 in medium 1 and v_2 in medium 2, where

$$\begin{aligned}v_1 &= \frac{c}{n_1}, \\ v_2 &= \frac{c}{n_2}.\end{aligned} \tag{3.10}$$

Assuming that the light ray left point P at time $t = 0$, we can see from the figure that the time t it takes for light ray to arrive point Q is given by

$$t = \frac{r_1}{v_1} + \frac{r_2}{v_2} = \frac{n_1\sqrt{a^2 + x^2}}{c} + \frac{n_2\sqrt{b^2 + (d - x)^2}}{c}. \tag{3.11}$$

To find the trajectory such that the time of arrival is minimum, we need to differentiate the above equation (of course with respect to x, which is the only variable upon which t is dependent), and set it to 0. It is shown is the following steps

$$\frac{dt}{dx} = \frac{n_1 x}{c\sqrt{a^2 + x^2}} - \frac{n_2(d - x)}{c\sqrt{b^2 + (d - x)^2}} = 0, \tag{3.12}$$

which implies

$$\frac{n_1 x}{c\sqrt{a^2 + x^2}} = \frac{n_2(d - x)}{c\sqrt{b^2 + (d - x)^2}}. \tag{3.13}$$

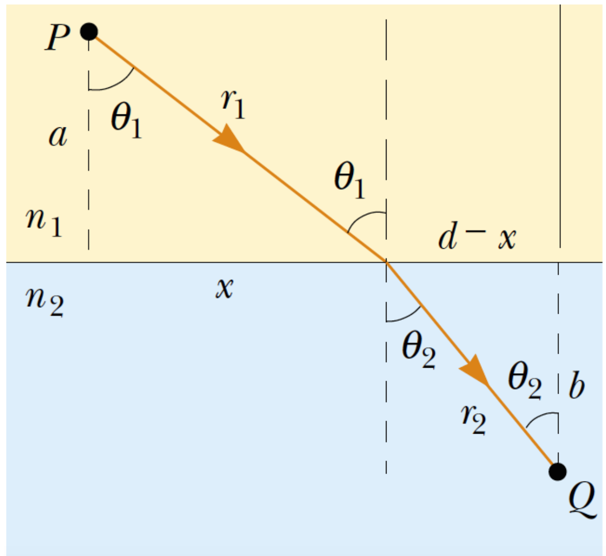

Figure 3.1: Geometry for deriving Snells law of refraction using Fermats principle.

Substituting,

$$\begin{aligned} \sin\theta_1 &= \frac{x}{\sqrt{a^2+x^2}}, \\ \sin\theta_1 &= \frac{d-x}{\sqrt{b^2+(d-x)^2}} \end{aligned} \tag{3.14}$$

in equation (13) finally yields,

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2.} \tag{3.15}$$

Practice Problem: Derive law of reflection using Fermat's principle.

3.3 Electrostatics: Method of Images

In Electrostatics, some problems involving conductors and charges can be solved with a great simplicity if the shapes of conductors are special, and has some special symmetry. In this case, the induced charges on the conductor can be replaced by some finite image charges. The main idea is that the potential due to charge and its image at the conductor's surface must give a constant. This concept will be demonstrated in the following problem (This problem partly appeared in IPhO 2010)

3.3.1 Charge outside a gounded Spherical Conductor

Consider a grounded conducting sphere of radius R located at a distance d from a point charge q (the distance is measured center of the sphere). To find the potential of this configuration, we replace the induced surface charges on the sphere by an image charge q' at a distance $d' < R$ inside the sphere and along the line connecting the center of sphere and charge q. Note that there is no obvious proof that this method works. One simply postulates an image charge with certain value at a certain position and then test whether or not obtained values of charge and distance can make up a constant potential at the surface of the sphere.

Since the sphere is conducting and grounded $V = 0$ through out the surface of the sphere. Using this we get

$$V(P) = \frac{q}{4\pi\epsilon_0(d-R)} + \frac{q'}{4\pi\epsilon_0(R-d')} = 0, \tag{3.16}$$

and

$$V(Q) = \frac{q}{4\pi\epsilon_0(d+R)} + \frac{q'}{4\pi\epsilon_0(R+d')} = 0. \tag{3.17}$$

The above equations give

$$\frac{d+R}{d-R} = \frac{R+d'}{R-d'}, \tag{3.18}$$

which upon simplification gives

$$\boxed{d' = \frac{R^2}{d}.} \tag{3.19}$$

Using above equation in (3.16) gives

$$\boxed{q' = -q\frac{R}{d}.} \tag{3.20}$$

This finally gives

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{R/d}{\left(r^4+\frac{R^4}{d^2}-2\frac{a^2}{d}r\cos\theta\right)^{1/2}}\right]. \tag{3.21}$$

If you want to understand all sorts of interesting physics related to the image problem, I definitely recommend you to go through the full problem from IPhO 2010. I find the last part of that problem - oscillation of charged pendulum-fun.

Chapter 4

Special Theory of Relativity

The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties . . . .

The twentieth century experienced for the first time two revolutionary concepts of physics. First is the Theory of Relativity (both special and general), and the second is Quantum Physics. While Einstein solely formulated the theory of Relativity, several people (including Einstein) have importantly contributed in the field of Quantum Physics. Both of these concepts challenge our ordinary way of thinking, and are oftentimes not so easy to visualise. In this section, we will discuss the Special Theory of Relativity. This theory covers phenomena such as the slowing down of moving clocks and the contraction of moving lengths. We also discuss the relativistic forms of momentum and energy.

4.1 Postulates of Special Relativity

The two basic postulates are
1. Physics is the same in all inertial frames.
2. The speed of light is the same in all inertial frames.

The first postulate is just the principle of Galilean relativity, whose implication is that there is no absolute frame of reference; every inertial reference frame is as good as every other one.

The second postulate is the heart of special theory of relativity, as all the strange (or nonintuitive) physics depicted by special relativity is because of the fact the particles cannot travel faster than the speed of light. To some extent, the second postulate is a corollary of the first once one realizes that electromagnetic waves do not travel in a medium. If the laws of electromagnetism, which give rise to the speed of light, are to be the same in all frames, then the speed of light must of necessity be the same in all frames.

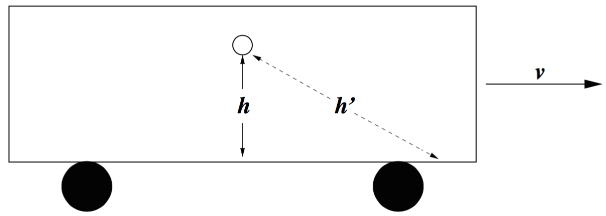

Figure 4.1: Geometry for deriving Time Dilation.

4.2 Time Dilation

The basic premise for time dilation is that two events that are simultaneous in one reference frame are, in general, not simultaneous in a second frame moving relative to the first. That is, simultaneity is not absolute but rather depends on the state of motion of the observer.

Consider a pulse of light emitted from a source located at the center of a train, which is moving at speed v in the x direction with respect to a station (Ref. 4.1). The pulse of light is directed toward a photosensor on the floor of the train, directly under the bulb. We are interested in the time interval between the emission of photon and the detection of it at the floor.

The time interval measured depends on who measures it. An observer at ground and an observer at train will have a different measurements. In the frame of train, the light pulse travels distance h before being detected. Therefore, time interval Δt_0 is

$$\Delta t_0 = \frac{h}{c}. \tag{4.1}$$

Let the time interval in ground frame be Δt. Then the ground observer claims that photon not only travel h vertically, but also travel a horizontal distance of $v\Delta t$ before being detected. Therefore, in the ground frame, the photon travels a distance of $h' = \sqrt{h^2 + (v\Delta t)^2}$ before being detected, which implies

$$t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c}. \tag{4.2}$$

Using (16) and (17) gives

$$c^2(\Delta t)^2 = c^2(\Delta t_0)^2 + v^2(\Delta t)^2,$$

which finally gives

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \equiv \gamma \Delta t_0}, \tag{4.3}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{4.4}$$

Since $\gamma \geq 1$, it is easy to deduce from the above equation that moving clock runs slow.

Problem: The average lifetime of a π meson in its own frame of reference is 26.0 ns. (a) If the π meson moves with speed 0.95c with respect to the Earth, what is its lifetime as measured by an observer at rest on Earth? (b) What is the average distance it travels before decaying as measured by an observer at rest on Earth?

Solution: The frame of muon is the proper rest frame, and therefore its measured life-time is the proper life-time. The life-time measure by an observer at rest on Earth is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = 83.3\text{ns},$$

and therefore the distance it travels as measured by an observer at rest on Earth is

$$L = v\Delta t = 24.0m$$

Problem: An atomic clock is placed in a jet airplane. The clock measures a time interval of 3600 s when the jet moves with speed 400 m/s. How much larger a time interval does an identical clock held by an observer at rest on the ground measure?

Solution: The proper time is the time measured by the clock placed in the jet. Quantity we are interested in is

$$\begin{aligned}
\Delta t - \Delta t_0 &= (\gamma - 1)\Delta t_0 \\
&\approx (1 + \frac{v^2}{2c^2} - 1)\Delta t_0 \\
&= \frac{v^2}{2c^2}\Delta t_0.
\end{aligned}$$

Note that the approximation above was absolutely necessary as $\gamma \approx 1$, and your calculator will probably show $\gamma = 1$, which will give you a result 0, which is not what we want here. Substituting values in the above equation gives 3.2 ns which is so much smaller than the time involved in the problem. This problem illustrates that time dilation is negligible for low speeds.

4.3 Length Contraction

Imagine the same train as shown in figure (4.1), which flashes light from the left end towards the right end. The right end constains a perfectly reflecting mirror, and light reflects back to the left end after it striking the mirror at right end. If the time interval of this event (emission of light and reception of it) is Δt_0, we can infer that the length of train Δx_0, in the train's frame is

$$\Delta x_0 = \frac{c\Delta t_0}{2}. \tag{4.5}$$

In the ground frame, let the time interval from emmision of light to its arrival at right end be Δt_1, and let the time interval from the reflection at right end to its reception at right end be Δt_1. The the following holds:

$$\begin{aligned} c\Delta t_1 &= \Delta x_0 + v\Delta t_1 \\ c\Delta t_2 &= \Delta x_0 - v\Delta t_1, \end{aligned} \tag{4.6}$$

which gives

$$\begin{aligned} \Delta t_1 &= \frac{\Delta x_0}{c+v} \\ \Delta t_2 &= \frac{\Delta x_0}{c-v}, \end{aligned} \tag{4.7}$$

which makes sense because in the first trip, relative velocity of train and light is smaller while in the second trip, it is larger.Therfore, the total interval of this event in the ground frame is

$$\begin{aligned} \Delta t &= \Delta t_1 + \Delta t_2 \\ &= \frac{\Delta x_0}{c+v} + \frac{\Delta x_0}{c-v} \\ &= \frac{2\Delta x_0}{c\left(1 - v^2/c^2\right)} \\ &= \frac{2\gamma^2 \Delta x_0}{c}. \end{aligned} \tag{4.8}$$

As $\Delta t = \gamma \Delta t_0$, we finally get

$$\boxed{\Delta x = \frac{\Delta x_0}{\gamma}}, \tag{4.9}$$

where Δx is the length measured by the observer at ground level. Since $\gamma \geq 1$, moving objects get contracted.Note that the length parallel to the motion of an object is contracted.

Problem: A rod of length L_0 moves with speed v along the horizontal direction. The rod makes an angle θ_0 with respect to the x axis. (a) Determine the length of the rod as measured by a stationary observer. (b) Determine the angle θ the rod makes with the x' axis.

Solution The key to solve this problem is to note that only the length along the direction of velocity is contracted. As $L_x = L_0 \cos\theta$ and $L_y = L_0 \sin\theta$, we get

$$\begin{aligned} L'_{x'} &= \frac{L_0 \cos\theta_0}{\gamma} \\ L'_{y'} &= L_0 \sin\theta_0. \end{aligned}$$

which gives the length in the reference frame of stationary observer as

$$L' = \sqrt{\left(\frac{L_0 \cos\theta_0}{\gamma}\right)^2 + (L_0 \sin\theta_0)^2} = L_0 \sqrt{1 - \left(\frac{v^2}{c^2}\right)\cos^2\theta_0}.$$

The angle made by the rod with x' is given by

$$\tan\theta = L'_{y'}/L'_{x'} = \gamma \tan\theta_0.$$

4.3.1 Why is there no length in the direction perpendicular to the motion?[1]

A very simple thought experiment suggested by Taylor and Wheeler[2] can be used to show that length contraction can't happen in directions perpendicular to the direction of motion. Imagine that we build a wall beside the railroad tracks, and 1 m above the rails, as measured on the ground, we paint a horizontal blue line. When the train goes by, a passenger leans out the window holding a wet paintbrush 1 m above the rails, as measured on the train, leaving a horizontal red line on the wall. Question: Does the passenger s red line lie above or below our blue one? If the rule were that perpendicular directions contract, then the person on the ground would predict that the red line is lower, while the person on the train would say it s the blue one (to the latter, of course, the ground is moving). The principle of relativity says that both observers are equally justified, but they cannot both be right. No subtleties of simultaneity or synchronization can rationalize this contradiction; either the blue line is higher or the red one is–unless they exactly coincide which is the inescapable conclusion. There cannot be a law of contraction (or expansion) of perpendicular dimensions, for it would lead to irreconcilably inconsistent predictions.

4.4 Energy Momentum Relations

The classical formulas for kinetic energies do not work well in special relativity. Details of the energy momentum relations will not be included in this section. Only relevant equations will be presented here. The momentum of a particle of mass m moving at a speed v is

$$\boxed{p = \gamma m v}, \tag{4.10}$$

and the energy relation is

$$\boxed{E = \gamma m c^2.} \tag{4.11}$$

Using the above two equations, it can be shown that

$$\boxed{E = \sqrt{p^2c^2 + m^2c^4}.} \tag{4.12}$$

Note that p takes classical form when $\gamma \sim 1$, whereas it differs significantly at speeds $v \sim c$. Moreover, energy of a rest particle ($\gamma = 1$) is mc^2. Therefore, we

[1]Source: Griffiths, David Jeffrey, and Reed College. Introduction to electrodynamics. Vol. 3. Upper Saddle River, NJ: prentice Hall, 1999.

[2]E. F. Taylor and J. A. Wheeler, Spacetime Physics (San Francisco: W. H. Freeman, 1966)

define kinetic energy as

$$T = E - mc^2 = (\gamma - 1)mc^2. \tag{4.13}$$

Using binomial expansion for $v \ll c$, i.e $v/c \ll 1$, we can write,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{v^2}{2c^2}, \tag{4.14}$$

which implies,

$$T \approx (1 + \frac{v^2}{2c^2} - 1)mc^2 = \frac{1}{2}mv^2. \tag{4.15}$$

The above equation shows that at low speeds, Newtonian results and results from relativity matches. Since, we have to deal mostly with low speeds in our everyday life, Newtonian mechanics seems to work so well.

Problem: A photon of wavelength λ strikes an electron, initially at rest. After the collision the photon travels at angle ϕ relative to its original direction. Find the new wavelegth λ' of the photon after the collision. This phenomenon is called Compton Effect.

Solution We need to use relativistic equations,as the problem involves electron, which is very light and usually moves at large speed. The key to solve this problem is to use conservation of energy and momentum.

If the momentum of photon before and after collision is $\mathbf{P_{ph}}$ and $\mathbf{P'_{ph}}$ and the momentum of electron after collision is $\mathbf{P_e}$, then using $\mathbf{P_{ph}} = \mathbf{P'_{ph}} + \mathbf{P_e}$ gives

$$P_e^2 = P_{ph}^2 + P'^2_{ph} - 2P_{ph}P'_{ph}\cos\phi,$$

while conservation of energy gives

$$P_{ph}c + mc^2 = P'_{ph}c + \sqrt{P_e^2c^2 + m_e^2c^4}.$$

Taking squuare root on one side and squaring the above equation, and finally subtracting it from the momentum conservation equation gives

$$mc\left(\frac{1}{P'_{ph}} - \frac{1}{P_{ph}}\right). \tag{4.16}$$

Finally, using $P_{ph} = h/\lambda$ and $P'_{ph} = h/\lambda'$ gives

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\phi).$$

This phenomenon is called Compton Scattering.

Self- Help Problem: Solve the same problem given above, this time using Newtonian equations.

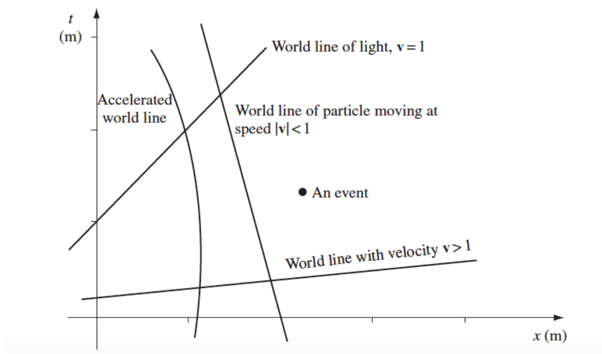

Figure 4.2: A spacetime diagram in natural units (i.e, c=1).

4.5 Lorenz Transformation

Imagine a particle at position x at time t measured at a rest frame. If the axis is moved towards right by a velocity v, then according to Gallelion transformation, the position x' and time t' measured with respect to moving axis is given by

$$x' = x - vt, \tag{4.17}$$

and

$$t' = t. \tag{4.18}$$

However, the second equation is already in trouble if we consider relativistic effects. The correct tranformation laws called Lorenz transformation are

$$\boxed{x' = \gamma(x - vt)}, \tag{4.19}$$

and

$$\boxed{t' = \gamma(t - vx/c^2).} \tag{4.20}$$

4.6 Spacetime Diagrams[3]

A very important part of learning the geometrical approach to SR is mastering the spacetime diagram. Figure (4.2) shows a two dimensional slice of spacetime, the tx plane, in which are illustrated the basic concepts. A single point in this space is a point of fixed x and fixed t, and is called an event. A line in the space gives a relation $x = x(t)$, and so can represent the position of a particle at different times. This is called the particles world line. Its slope is related to

[3]This section is taken from Schutz, Bernard. A first course in general relativity. Cambridge university press, 2009.

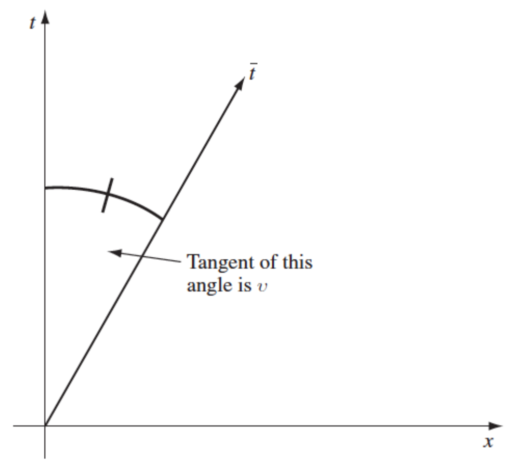

Figure 4.3: The time-axis of a frame whose velocity is v.

its velocity,

$$\text{slope} = \frac{dt}{dx} = \frac{1}{v}. \tag{4.21}$$

This concept is illustrated in figure (4.2). Notice that a light ray (photon) always travels on a 45° line in this diagram.

Since any observer is simply a coordinate system for spacetime, and since all observers look at the same events (the same spacetime), it should be possible to draw the coordinate lines of one observer on the spacetime diagram drawn by another observer. To do this we have to make use of the postulates of SR. Suppose an observer O uses the coordinates t, x as above, and that another observer $\tilde{O}$, with coordinates $\tilde{t}$, $\tilde{x}$, is moving with velocity v in the x direction relative to O. Where do the coordinate axes for $\tilde{t}$ and $\tilde{x}$ go in the spacetime diagram of O? $\tilde{t}$ axis: This is the locus of events at constant $\tilde{x} = 0$ (and $\tilde{y} = \tilde{z} = 0$, too, but we shall ignore them here), which is the locus of the origin of $\tilde{O}$s spatial coordinates. This is $\tilde{O}$s world line, and it looks like that shown in Fig. (4.3). $\tilde{x}$ axis: To locate this we make a construction designed to determine the locus of events at $\tilde{t} = 0$, i.e. those that $\tilde{O}$ measures to be simultaneous with the event $\tilde{t} = \tilde{x} = 0$. Consider the picture in $\tilde{O}$ s spacetime diagram, shown in Fig. (4.4).

The events on the $\tilde{x}$ axis all have the following property: A light ray emitted at event E from $\tilde{x} = 0$ at, say, time $\tilde{t} = a$ will reach the $\tilde{x}$ axis at $\tilde{t} = a$ (we call this event P); if reflected, it will return to the point $\tilde{t} = 0$ at $\tilde{t} = +a$, called event R. The $\tilde{x}$ axis can be defined, therefore, as the locus of events that reflect light rays in such a manner that they return to the $\tilde{t}$ axis at $+a$ if they left it at a, for any a. Now look at this in the spacetime diagram of O, Fig. (4.5).

We know where the $\tilde{t}$ axis lies, since we constructed it in Fig. (4.3). The events of emission and reception, $\tilde{t} = -a$ and $\tilde{t} = a$, are shown in Fig.(4.5).

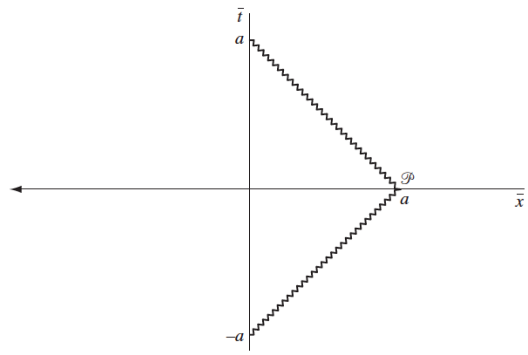

Figure 4.4: Light reflected at a, as measured by $\tilde{O}$.

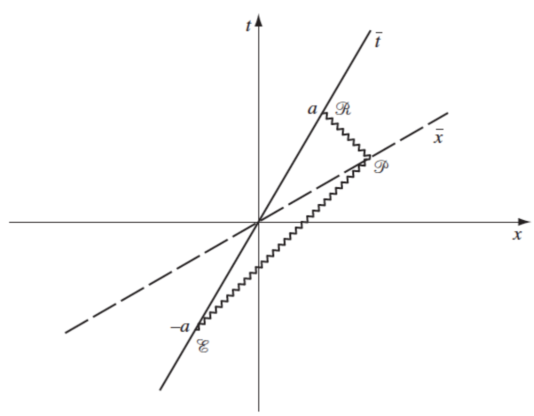

Figure 4.5: The reflection in Fig.(4.4), as measured O.

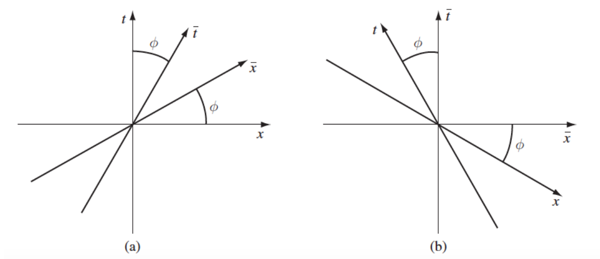

Figure 4.6: Spacetime diagrams of O (left) and $\tilde{O}$ (right).

Since a is arbitrary, it does not matter where along the negative $\tilde{t}$ axis we place event E, so no assumption need yet be made about the calibration of the $\tilde{t}$ axis relative to the t axis. All that matters for the moment is that the event R on the $\tilde{t}$ axis must be as far from the origin as event E. Having drawn them in Fig. (4.5), we next draw in the same light beam as before, emitted from E, and travelling on a 45° line in this diagram. The reflected light beam must arrive at R,so it is the 45° line with negative slope through R. The intersection of these two light beams must be the event of reflection P. This establishes the location of P in our diagram. The line joining it with the origin the dashed line must be the $\tilde{x}$ axis: it does not coincide with the x axis. If you compare this diagram with the previous one, you will see why: in both diagrams light moves on a 45° line, while the t and $\tilde{t}$ axes change slope from one diagram to the other. This is the embodiment of the second fundamental postulate of SR: that the light beam in question has speed $c = 1$ (and hence slope=1) with respect to every observer. When we apply this to these geometrical constructions we immediately find that the events simultaneous to $\tilde{O}$ (the line $\tilde{t} = 0$, his x axis) are not simultaneous to O (are not parallel to the line $t = 0$, the x axis). This failure of simultaneity is inescapable. The following diagrams (Fig. 4.6) represent the same physical situation. The one on the left is the spacetime diagram O, in which $\tilde{O}$ moves to the right.

The one on the right is drawn from the point of view of $\tilde{O}$, in which O moves to the left. The four angles are all equal to arctan $|v|$, where $|v|$ is the relative speed of O and $\tilde{O}$.

Practice Problems

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Problem 1: Levitation of Conducting Loop

A long horizontal wire carries a current I (towards right) which decreases with time. A conducting loop is suspended over a small time interval t. In that interval the loop is in a balanced position. The coil is located in a plane vertically at a distance D below the wire, as shown in Fig. The loop is a square of side a, mass m and resistance R. The distance D is much greater than a. Neglect the self-inductance of the coil, and define acceleration due to gravitiy as g.

1. Make a diagram of the system clearly indicating the currents, fields and magnetic forces involved.

   [2 points]

2. Using appropriate approximations, find the current induced in the loop, i, as a function of $\Delta I/\Delta t$.

   [3 points]

3. Find the net magnetic force on the loop, including its magnitude and direction.

   [3 points]

4. Calculate the condition that the product $I\frac{\Delta I}{\Delta t}$ must satisfy according to electromagnetic quantities given and constant, so that the loop is maintained levitating (or floating). Be very careful with the signs of the amounts involved.

   [2 points]

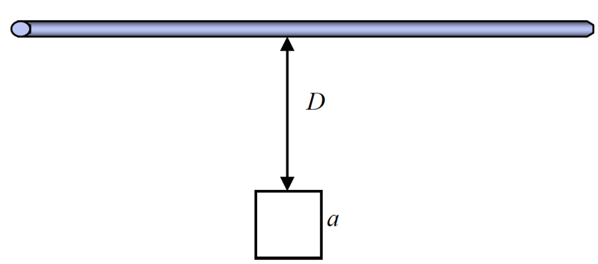

Figure 1: Set up of the Problem.

Useful relation:

$$\frac{1}{(1+x)^n} \approx 1 - nx,$$

when $|x| \ll 1$.

Problem 3: Rolling Cylinder

A uniform solid cylinder of mass M and radius a is set in rotation about its axis with an angular velocity ω_0. The cylinder is then lowered with its lateral surface onto a horizontal plane and released. The coefficient of friction between the cylinder and the plane is equal to μ.

1. Draw a free body diagram showing all forces involved in this problem after the cylider is released onto a horizontal plane. Is there any point in the cylinder about which the torque is zero? Is there any point in the cylinder about which the angular momentum is conserved?

   [2 points]

2. Write down the conditions for sliding and pure rolling. Your conditions should involve SOME of the following parameters: center of mass velocity of the cylinder v, angluar velocity ω, a, M and μ.

   [2 points]

3. Find the time t_0 upto which the cylinder will move with sliding.

   [3 points]

4. Find the total work performed by the sliding friction force acting on the cylinder.

   [3 points]