Errors and their Propagation

What is error/uncertainty?

As it may sound, error does not necessarily mean the mistake made by the experimenter. There is, in fact, always a limitation as to how precise or exact a measurement can be. There will always be an uncertainty associated with a measured quantity (error and uncertainty basically mean the same thing). Therefore, we need to consider those errors in our experiment to deduce a reasonable conclusion.

An uncertainty in a quantity x is denoted by Δx or δx.

Types of errors on the basis of origin:

1. **Errors in Direct Measurements**
   When you measure something directly, like distance and time with a measuring scale and stopwatch, there is some uncertainty due to the precision of the measuring device. Generally, the uncertainty in a direct measurement is half the value of Least Count (L.C.) of the measuring instrument.

   $$\Delta x = \frac{1}{2} \text{ L.C.}$$

2. **Errors in Indirect Measurements**
   When you measure something by measuring something else, like measuring velocity by using the distance and time measured directly, it is indirect measurement.
   The errors in directly measured quantities lead to errors in the indirectly measured quantity too. This is called error propagation, and it is very important to know how these errors propagate in a calculation.

Rules of Error Propagation:

Before knowing the rules of error propagation, keep in mind that

- uncertainty is rounded to one significant figure,
- the best estimate and uncertainty must always have the same number of digits after the decimal point, and
- you should *not* round in between calculation of error – keep some more significant figures until you find the error.

eg: (2 ± 0.01) cm and (3.00 ± 1) cm are incorrect as the number of digits after the decimal point of the quantity and its uncertainty do not match. (5.10 ± 0.20) cm is incorrect as the uncertainty has two significant figures – its correct form is (5.1 ± 0.2) cm.

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Rules:

- In addition or subtraction, the errors "always add."
  If $z = w + x - y$, and Δw, Δx, and Δy are the uncertainties in w, x, and y, then the uncertainty in z is,
  $$\Delta z = \Delta w + \Delta x + \Delta y$$

  eg: $w = (4.52 \pm 0.02)$ cm, $x = (2.0 \pm 0.2)$ cm, $y = (3.0 \pm 0.6)$ cm. Find $z = x + y - w$ and its uncertainty.

  $z = x + y - w = 2.0 + 3.0 - 4.52 = 0.48$ cm
  $\Delta z = \Delta x + \Delta y + \Delta w = 0.2 + 0.6 + 0.02 = 0.82 \simeq 0.8$ cm.

  The final answer, after rounding, is (0.5 ± 0.8) cm.

- When we divide or multiply two measured quantities x and y, the fractional uncertainty in the answer is the sum of the fractional uncertainties in x and y.
  If $z = xy$ or $z = x/y$,
  $$\frac{\Delta z}{|z|} = \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|}$$

  Note: It is necessary to take the absolute value as a negative value might reduce the uncertainty, but uncertainties are always added.

  eg: The masses of two bodies are $m_1 = (2.4 \pm 0.2)$ kg and $m_2 = (1.0 \pm 0.1)$ kg. Calculate their reduced mass, $\mu = \frac{m_1 m_2}{m_1 + m_2}$, and its uncertainty.

  $\mu = \frac{2.4 \times 1.0}{2.4 + 1.0} = 0.706$ kg (one more significant digit is kept)

  First, fractional uncertainty in the numerator is
  $$\frac{\Delta(m_1 m_2)}{m_1 m_2} = \frac{\Delta(m_1)}{m_1} + \frac{\Delta(m_2)}{m_2} = \frac{0.2}{2.4} + \frac{0.1}{1.0} = 0.183$$

  Second, fractional uncertainty in the denominator is
  $$\frac{\Delta(m_1 + m_2)}{m_1 + m_2} = \frac{\Delta m_1 + \Delta m_2}{m_1 + m_2} = \frac{0.2 + 0.1}{2.4 + 1.0} = 0.088$$

  Now, $\frac{\Delta\mu}{\mu} = \frac{\Delta(m_1 m_2)}{m_1 m_2} + \frac{\Delta(m_1 + m_2)}{m_1 + m_2} = 0.183 + 0.088 = 0.271$

  ⇨ $\Delta\mu = 0.706 \times 0.271 = 0.19$ kg $\simeq 0.2$ kg (rounded to one significant figure)

  Thus, the final answer is (0.7 ± 0.2) kg.

- If $z = x^m y^n$ or x^m/y^n,

  $$\frac{\Delta z}{|z|} = |m| \frac{\Delta x}{|x|} + |n| \frac{\Delta y}{|y|}$$

- If $z = ax$, where a is an exact number like G, N_A, k_B, R, or π,

  $$\Delta z = a\, \Delta x$$

In general, the uncertainty of a function can be found by taking its derivative, except that the *negative terms are made positive*.

eg: If $z = \frac{x}{y}$, $dz = \frac{y\,dx - x\,dy}{y^2}$

$$\frac{dz}{z} = \frac{dz}{(x/y)} = \frac{dx}{x} - \frac{dy}{y}$$

So, after changing the negative sign to positive, $\frac{\Delta z}{|z|} = \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|}$

Some more examples:

- If $z = \sqrt{x^2 + y^2}$, $dz = d(x^2 + y^2)^{1/2} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x\,dx + 2y\,dy)$

  ⇨ $\Delta z = \frac{x\,\Delta x + y\,\Delta y}{\sqrt{x^2+y^2}}$

- If $z = \log_e x$, $\frac{dz}{dx} = \frac{d(\log_e x)}{dx} = \frac{1}{x}$

  ⇨ $\Delta z = \frac{1}{x}\Delta x$

- If $z = e^x$, $\frac{dz}{dx} = \frac{d(e^x)}{dx} = e^x$

  ⇨ $\Delta z = e^x \Delta x$

- If $z = \sin\theta$, $\frac{dz}{d\theta} = \frac{d\sin\theta}{d\theta} = \cos\theta$

  ⇨ $\Delta z = \cos\theta\, \Delta\theta$

  Here, θ must be in radians.

Exercise:

A) Compute the following:

a) $(5 \pm 1) + (8 \pm 2) - (10 \pm 4)$
b) $(3.5 \pm 0.1) \times (8.0 \pm 0.2)$
c) $(10 \pm 1)/(20 \pm 2)$
d) $(3.5 \pm 0.1) \times (8.0 \pm 0.2)/(5.0 \pm 0.4)$

B) A student makes the following measurements:
$a = (5 \pm 1)$ cm, $b = (18 \pm 2)$ cm, $c = (12 \pm 1)$ cm, $t = (3.0 \pm 0.5)$ s, $m = (18 \pm 1)$ gram.

Compute the following quantities with their uncertainties and percentage uncertainties:

e) $a + b + c$
f) $a + b - c$
g) ct
h) mb/t

C) If I have measured the radius of a sphere as $r = (2.0 \pm 0.1)$ m, what should I report for the sphere's

i) area
j) volume

D) The voltage across a capacitor is found to decay with time according to the relation

$$V = V_0 e^{(-t/T)}$$

If $V = (1.8 \pm 0.1)$ volt, $V_0 = (4.3 \pm 0.1)$ volt, and $t = (2.00 \pm 0.05)$ s, find T.

Answers:

A.
a) 3 ± 7
b) 28 ± 2
c) 0.5 ± 0.1
d) 5.6 ± 0.7

B.
e) (35 ± 4) cm = 35 cm $\pm$ 10%
f) (11 ± 4) cm = 11 cm $\pm$ 40%
g) (36 ± 9) cm = 36 cm $\pm$ 25%
h) (110 ± 40) grams·cm/s = 110 grams·cm/s $\pm$ 30%

C.
i) (50 ± 5) m^2
j) (34 ± 5) m^3

D. (2.3 ± 0.2) s

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